Calculus for the Biological Sciences

Lecture Notes – Linear Models

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Outline

1. Linear Models
   - Chirping Crickets and Temperature
2. Equation of Lines
   - Slope-Intercept
   - Point-Slope
   - Two Points - Slope
   - Parallel and Perpendicular Lines
   - Intersection of Lines
3. Inverse Linear Function
4. Other Linear Models

Cricket Equation as a Linear Model

The line creates a mathematical model

- The temperature, T as a function of the rate snowy tree crickets chirp, Chirp Rate, N

There are several Biological and Mathematical questions about this Linear Cricket Model

There is a complex relationship between the biology of the problem and the mathematical model

Biological Questions – Cricket Model

When can this model be applied from a practical perspective?

- Biological thermometer has limited use
- Snowy tree crickets only chirp for a couple months of the year and mostly at night
- Temperature needs to be above 50°F

Mathematical Questions – Cricket Model

Over what range of temperatures is this model valid?

- Biologically, observations are mostly between 50°F and 85°F
- Thus, limited range of temperatures, so limited range on the Linear Model
- Range of Linear functions affects its Domain
- From the graph, Domain is approximately 50–200 Chirps/min

Mathematical Questions – Cricket Model

How accurate is the model and how might the accuracy be improved?

- Data closely surrounds Bessey Model –No more than about 3°F away fom line
- Dolbear Model is fairly close though not as accurate – Sufficient for rapid temperature estimate
- Observe that the temperature data trends lower at higher chirp rates – compared against linear model
- Better fit with Quadratic function –Is this really significant?

Equation of Line – Slope-Intercept Form

The Slope-Intercept form of the Line

$$y = mx + b$$

- The variable x is the independent variable
- The variable y is the dependent variable
- The slope is m
- The y-intercept is b

Equation of Line – Cricket-Thermometer

The folk/Dolbear model for the cricket thermometer

$$T = \frac{N}{4} + 40$$

- The independent variable is N , chirps/min The
- dependent variable is T , the temperature
- Thus, the temperature can be estimated from counting the number of chirps/min
- Equivalently, the temperature (measurement) depends on how rapidly the cricket is chirping

Equation of Line – Point-Slope Form

The Point-Slope form of the Line is often the most useful form

$$y - y_0 = m(x - x_0)$$

or

$$y = m(x - x_0) + y_0$$

- The slope is m
- The given point is (x_0, y_0)
- Again the independent variable is x, and the dependent variable is y

Equation of Line – Two Points

Given two points (x_0, y_0) and (x_1, y_1), the slope is given by

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

Use the previous point-slope form of the line satisfies

$$y = m(x - x_0) + y_0$$

where the slope is calculated above and either point can be used.

Example – Slope and Point

Find the equation of a line with a slope of 2, passing through the point $(3, -2)$.
What is the y-intercept?

Example

The point-slope form of the equation gives:

$$\begin{aligned} y - (-2) &= 2(x - 3) \\ y + 2 &= 2x - 6 \\ y &= 2x - 8 \end{aligned}$$

Example – Two Points

Find the equation of a line passing through the points $(-2, 1)$ and $(3, -2)$

Example

The slope satisfies

$$m = \frac{1-(-2)}{-2-3} = -\frac{3}{5}$$

From the point-slope form of the line equation, using the first point

$$y - 1 = -\frac{3}{5}(x+2)$$

$$y = -\frac{3}{5}x - \frac{1}{5}$$

Parallel and Perpendicular Lines

Consider two lines given by the equations:

$$y = m_1 x + b_1 \qquad \text{and} \qquad y = m_2 x + b_2$$

The two lines are parallel if the slopes are equal, so

$$m_1 = m_2$$

and the y-intercepts are different.

If $b_1 = b_2$, then the lines are the same.

The two lines are perpendicular if the slopes are negative reciprocals of each other, that is

$$m_1 m_2 = -1$$

Example – Perpendicular Lines

Find the equation of the line perpendicular to the line

$$5x + 3y = 6$$

passing through the point $(-2, 1)$

Example

Solution: The line can be written

$$3y = -5x + 6$$

$$y = -\frac{5}{3}x + 2$$

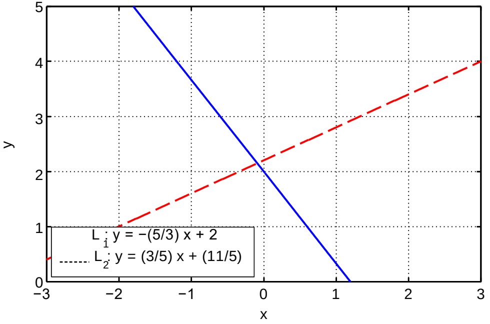
The slope of the perpendicular line (m_2) is the negative reciprocal

$$m_2 = \frac{3}{5}$$

Example – Perpendicular Lines

The point slope equation of the perpendicular line is

$$y - 1 = \frac{3}{5}(x + 2)$$
$$y = \frac{3}{5}x + \frac{11}{5}$$



Example – Intersection of Lines 1

Find the intersection of the line parallel to the line $y = 2x$ passing through $(1, -3)$ and the line given by the formula

$$3x + 2y = 5$$

Skip Example

Solution: The line parallel to $y = 2x$ has slope $m = 2$, so satisfies

$$\begin{aligned} y + 3 &= 2(x - 1) \\ y &= 2x - 5 \end{aligned}$$

Example – Intersection of Lines 2

Continued: Substitute y into the formula for the second line

$$3x + 2(2x - 5) = 5$$

$$7x = 15 \quad \text{or} \quad x = \frac{15}{7}$$

Substituting the x value into the first line equation gives

$$y = 2\,\frac{15}{7} - 5 = -\frac{5}{7}$$

The point of intersection is

$$(x, y) = \frac{15}{7}, -\frac{5}{7}$$

Example – Temperature 1

Convert Temperature Fahrenheit to Celsius

- The freezing point of water is 32°F and 0°C, so take

$$(f_0, c_0) = (32, 0)$$

- The boiling point of water is 212°F and 100°C (at sea level), so take

$$(f_1, c_1) = (212, 100)$$

Example – Temperature 2

Convert Temperature Fahrenheit to Celsius

Solution: The slope satisfies

$$m = \frac{c_1 - c_0}{f_1 - f_0} = \frac{100 - 0}{212 - 32} = \frac{5}{9}$$

The point-slope form of the line gives

$$c - 0 = \frac{5}{9}(f - 32)$$

$$c = \frac{5}{9}(f - 32)$$

The temperature f in Fahrenheit is the independent variable The equation of the line gives the dependent variable c in Celsius

Inverse Linear Function

Linear functions always have an Inverse
(Provided $m \neq 0$)

Consider the line

$$y = mx + b$$

Solving for x

$$\begin{aligned} mx &= y - b \\ x &= \frac{y - b}{m} \end{aligned}$$

The Inverse Line satisfies

$$x = \frac{1}{m} y - \frac{b}{m}$$

Example - Inverse Line

The equation for converting °F to °C is

$$c = \frac{5}{9}(f - 32)$$

So,

$$f - 32 = \frac{9}{5}c$$

The equation for converting °C to °F is

$$f = \frac{9}{5}c + 32$$

Scuba Diving Model 1

The pressure of air delivered by the regulator to a Scuba diver varies linearly with the depth of the water

The regulator delivers air to a Scuba diver as follows: Air

- Pressure at 29.4 psi when at 33 ft
- Air Pressure at 44.1 psi when at 66 ft

Find the pressure of air delivered at the surface (0 ft.), at 50 ft., and at 130 ft. (the maximum depth for recreational diving).

Scuba Diving Model – Solution 2

Solution: The linear model

$$p = md + p_0$$

where p is the pressure (psi) and d is the depth in feet The data give two points

$$(d_0, p_0) = (33, 29.4) \quad \text{and} \quad (d_1, p_1) = (66, 44.1)$$

The slope is

$$m = \frac{44.1 - 29.4}{66 - 33} = \frac{14.7}{33} = 0.445 \text{ psi/ft}$$

Solution (cont): The linear model satisfies

$$p - 29.4 = 0.445(d - 33)$$
$$p = 0.445d + 14.7$$

- At the surface, $d = 0$ and the air pressure is 14.7 psi At a
- depth of $d = 50$ ft, the air pressure is 36.95 psi At a depth
- of $d = 130$ ft, the air pressure is 72.55 psi
- Note these assume we are at sea level and diving in sea water

Scuba Diving Model – Graph

Below is a graph of the data and the Linear Pressure Model